MUSICAL COMPOSITION WITH ZECKENDORF REPRESENTATIONS

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Abstract. Three contrasting polyphonic musical compositions based on Zeckendorf representations in the style of music characterized by Fibonacci numbers and the golden ratio are presented and analyzed.

In *A Style of Music Characterized by Fibonacci Numbers and the Golden Ratio* [1], presented at the 13th International Conference on Fibonacci Numbers and Their Applications (Patras, Greece, 2008), a tuning system based on the golden ratio and Fibonacci numbers was introduced. In addition, some general characteristics of a musical style were described and compositional methods were discussed using three simple monophonic compositions based on Fibonacci-related integer sequences as examples. In this paper, the Zeckendorf representations are shown to be a potentially interesting source for the composition of polyphonic music.

1. Introduction

Let $t \in \mathbb{N}^+$ and
$$\varphi = \frac{\sqrt{5}-1}{2}\ .$$
In the above mentioned article, a system of equally tempered musical scales was presented in which the unit interval of each tuning is $\varphi^t + 1$. Using this system of tunings as a basis, a style of music was introduced in which each work is a sonification of one or more mathematical sequences related to the Fibonacci numbers and the golden ratio. The sonification presents a sequence S in time-order, so that $S_i(n)$ is sonified at time n; the sequence values $S_i(n)$ determine the sonic properties of the note heard at n – musical parameters mapped to might include frequency, loudness, location etc. The score in each work includes a point graph of the sequence used as in the following, in which the unit interval is $\varphi^2 + 1$.

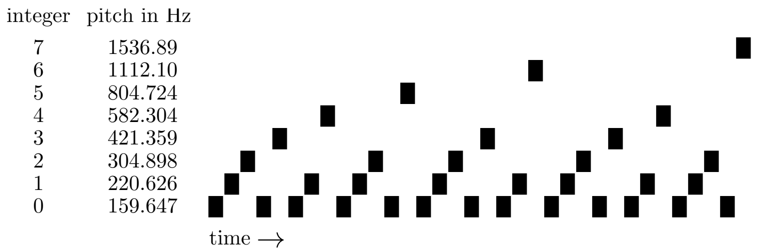


Figure 1; from *Horizontal Para-Fibonacci Sequence no. 1* (B5 in [2])

2. Characteristics of Compositions Based on Zeckendorf Representations

Although compositional choices such as tuning and tempo determine to a great extent the musical result of works based on Zeckendorf representations, as will be seen, it is indeed possible to make generalizations about the most interesting and prominent musical features of such compositions. These characteristics are directly related to some of the most fundamental properties of the mathematics of Zeckendorf representations.

2.1. **Mapping Zeckendorf Representations to Musical Parameters.**

Zeckendorf's theorem [3, 4, 5] guarantees $n \in \mathbb{N}^+$ can be represented as

$$Z(n) = \sum_{j=2}^{M} c_j F_j,$$

where $c_j \in \{0, 1\}$ and $c_j c_{j+1} = 0$. This Zeckendorf representation is minimal in that the least number of Fibonacci numbers (or bits) are used. We define three functions from this, using wedge brackets $<>$ to indicate a sequence and curly brackets $\{\}$ to indicate a set:

$$Z_b(n) = < c_j \mid F_j \in Z(n) >$$

$$Z_i(n) = \ \{j \mid F_j \in Z(n)\}$$

$$Z_s(n) = \ \{F_j \mid F_j \in Z(n)\}$$

Thus, for example,

$$Z(17) = 1F_7 + 0F_6 + 0F_5 + 1F_4 + 0F_3 + 1F_2.$$

$Z_b(17) \;= <1,0,0,1,0,1>$, usually written as the bit string 100101 – the sequence of coefficients c and conventional Zeckendorf representation of 17

$Z_i(17) \;=\; \{2,4,7\}$, the set of Fibonacci indices

$Z_s(17) \;=\; \{1,3,13\}$, the set of Fibonacci numbers summing to 17

$Z_i(n)$, or $Z_b(n)$ (where a black point indicates a 1), can be graphed as follows:

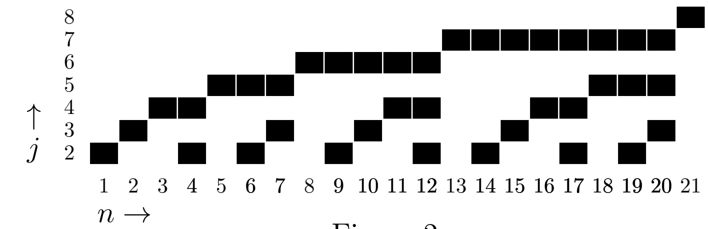


Figure 2

$Z_s(n)$ can be graphed as:

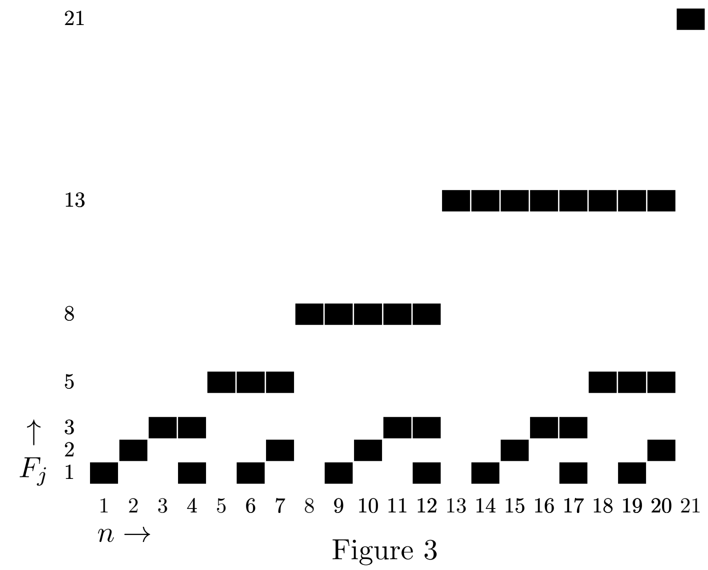


Figure 3

These two functions are suggestive of different compositional approaches to mapping the Zeckendorf representations to musical parameters; one can either choose to sonify the representations using the indices of the Fibonacci numbers $Z_i(n)$ or, alternatively, the Fibonacci numbers $Z_s(n)$. The utilization of both of these functions in compositions based on Zeckendorf representations will be discussed later.

2.2. **Where in the Sequence a Composition Terminates.**

The number of representations used in a composition plays a significant role in characterizing the resulting music. One can differentiate between two different types of composition in this style: finite and infinite. Finite works are those meant to be heard in their entirety and such works must, of course, have a point of termination. In contrast, inifinite compositions – which could, for example, be running continuously for long periods of time as installations in an art gallery – do not carry this limitation; these are concept works. In the case of finite compositions, the final representation heard should for aesthetic purposes satisfy $Z(F_n)$ or $Z(F_n - 1)$; in the former case the composition ends on a single note, in the latter it ends with a dense chord. The works discussed in this paper are exclusively finite.

2.3. **Seven Musically Perceivable Mathematical Properties of Zeckendorf Representations.**

As one becomes more familiar with the sound of compositions based on Zeckendorf representations and, in addition, those based on the Fibonacci numbers, the Wythoff array (A035513 in [6]), and the rabbit sequence (A003849 in [6]), also known as the golden string or infinite Fibonacci word, certain mathematical properties of the Zeckendorf representations become increasingly audible. In addition to familiarity with the mentioned sonified mathematical objects, the perceptibility of such phenomena is dependent on certain features of the composition being listened to. One important factor influencing the degree of perceptibility of a certain mathematical property is the choice of musical parameters mapped to; certain features might be perceived more clearly in one domain (e.g. pitch or loudness) than in another, and mapping to multiple domains in a single composition can increase the clarity of certain features, as will be shown. Mapping to the pitch domain can be particularly helpful in bringing out certain features. Another significant factor coming into play is tempo and, in Heisenbergian fashion, increasing or decreasing the tempo will tend to obscure one property while more clearly illuminating another. Some of the most musically relevant mathematical properties, all of which serve as inspiration in composition, are listed here in approximate order of how difficult they are to perceive in music, starting with perhaps the simplest.

2.3.1. *Sets Consisting of a Single Element at Fibonacci Indices.*
Because $Z_s(F_j) = \{F_j\}$ for $j \geq 2$, every time a Fibonacci number is sonified, only a single note is heard. The audibility of this phenomenon depends primarily on the speed with which the sequence is rendered; the slower the sequence is rendered, the more audible it becomes. If the representations are not mapped to the frequency domain, e.g. in compositions for unpitched percussion instruments, this feature becomes less audible.

2.3.2. *Greatest Integers Occurring in $Z_i(n)$ and $Z_s(n)$ Repeat According to the Fibonacci Sequence.*
Let the function $max[S]$ denote the greatest in a set of integers S. The sequence $< max[Z_i(n)] \,|\, n \in \mathbb{N}^+ >$ begins

$$< 2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, \cdots >,$$

which is A130233 in [6].
$< max[Z_s(n)] \,|\, n \in \mathbb{N}^+ >$ begins

$$< 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 8, 8, 13, 13, 13, 13, 13, 13, 13, 13, \cdots >,$$

A087172 in [6].

The lengths of the repetitions are also Fibonacci numbers. The perceptibility of this feature is less dependent on tempo, although extremely slow tempi will begin to obscure it.

2.3.3. *The Rabbit Sequence in the Zeckendorf Representations.*
In Figure 2, we see that in each row the blocks of 1s are of the same length; the blocks of 1s and 0s repeat according to the rabbit sequence. As a consequence of [7], the blocks of 1s in row j are of length F_{j-1}, and the blocks of 0s are of length F_j. In particular, we note the bottom row is the rabbit sequence itself $10010100\ldots$, A003849 in [6]. For row j, we use the substitution rules on the original rabbit sequence $1 \to 111\ldots1$ with F_{j-1} 1s and $0 \to 000\ldots0$ with F_j 0s. This property clearly becomes more audible with increased tempo and length of sequence, and mapping to the pitch domain will

help to more clearly illuminate it, although this feature is readily perceivable in multiple domains.

2.3.4. *The Relation Between the Wythoff Array and Zeckendorf Representations.*
There is an intimate relation between the horizontal para-Fibonacci sequence $P_h(n)$ (see A035612 in [6], which gives the column in which an integer n occurs in the Wythoff array) and $Z(n)$. So if we overlay a graph of the horizontal para-Fibonacci sequence $P_h(n)$ onto $Z_b(n)$, we notice that $P_h(n)$ defines the start of each block of 1s:

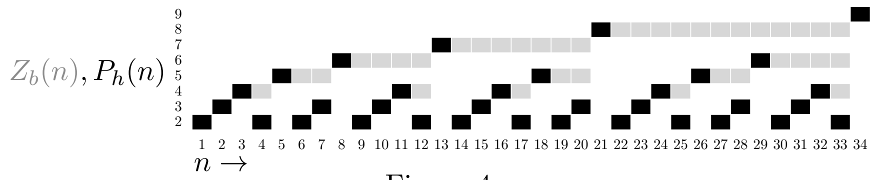


Figure 4

The audibility of this phenomenon is primarily dependent on one's familiarity with the sound of the horizontal para-Fibonacci sequence. Faster tempi will also reveal this feature more clearly.

2.3.5. *Self-Similarity.*
The sequence of Zeckendorf representations displays self-similarity; because $Z(n)$ is determined by a greedy algorithm,then if $x \in \mathbb{N}^+$ and $x < F_{j-1}$, the recursive relation $Z_s(F_j + x) = \{F_j\} \cup Z_s(x)$ holds for $j \geq 2$. Faster tempi reveal more clearly the recursive nature of music based on Zeckendorf representations – the effect of increasing the tempo and extending the sequence could be compared to the effect of increasing the resolution and number of iterations in a visual fractal.

2.3.6. *No Consecutive 1s in $Z_b(n)$.*
In the canonical binary representation $Z_b(n)$, no consecutive 1s occur. When mapping to the pitch domain, this becomes a harmonic property of the sequence of chords occuring in the piece. If one is mapping $Z_i(n)$, no chords containing two consecutive notes in the tuning occur; this feature is simpler to perceive if one is mapping $Z_i(n)$ rather than $Z_s(n)$, due to the Fibonacci interval spread in $Z_s(n)$ (compare Figure 2 to 3). It is also significantly easier to perceive at slower tempi, where relationships between the consecutive chords have less effect on harmonic perception.

2.3.7. *The Exhaustive Nature of Zeckendorf Representaitons.*
The principle of the exhaustion of a set of finite or infinite possibilities has been of artistic interest not only to Western classical musicians, but also to musicians of other cultures such as those of the Indian Carnatic classical music tradition; composer Kotisvara Ayer (1869-1938), for example, wrote in each of the 72 mela rāgas present in the Karnatak system [8]. Baroque composers such as Johann Sebastian Bach [9] and Johann Mattheson [10] took aesthetic interest in writing works in which each key existing in the system was methodically exhausted. Schoenberg's "method of composition with twelve tones related only to one another" [11] is an example of an entire method of composition in which this principle is paramount. The exhaustive nature of Zeckendorf representations make them an attractive and rich source of compositional material.

$Z_b(n)$ as n varies includes every possible bit string which does not include two consecutive 1s. Describing the musical experience of such an exhaustive property is about as difficult as describing the taste of a certain wine. In any case, there is a sense of completion upon hearing an entire set of defined finite possibilities. If the last representation heard is $Z(F_j - 1)$ in a finite composition, then every possible bit string containing no consecutive 1s up to this length occurs in the composition. The more representations used in a composition, the more difficult this exhaustive property is to perceive – the larger the set of possibilities, the more difficult it is to sense that all have occurred. The effect of tempo on the perception of this feature will be subject to future research.

2.4. **Tempo Limits and the Threshold of Perceptibility of Successively Played Tones.**

As Curtis Roads states in his book *Composing Electronic Music* [12], "When events fly by too quickly they 'blur' in our brain." The question of just how fast any sequence can be rendered so that each tone is perceived individually has been part of the author's work and is subject to further investigation. If one is mapping values to the frequency domain, it is clear that sequences can be rendered faster at higher frequencies because the brain needs somewhere between one and two full periods periods of a wave to perceive its fundamental frequency. Another phenomenon that deserves mention here is the fact that when one renders a sequence at more than around 20 values per second, a resultant pitch can begin to emerge – a pitch corresponding directly in frequency to the speed of rendering. This is due to the fact that the speed of rendering has entered the frequency range of human hearing, which spans from approximately 16 Hz to 20 kHz. If the attack – the initial onset – of the notes is strong, the resultant pitch becomes more prominent. Nonetheless, works such as B155 and B162 in [2] show that a speed of rendering of around 30 notes per second does not necessarily produce an audible resultant pitch when mapping to high frequencies (in the case of these two works, above 3212 Hz and 524 Hz respectively).

3. Three Contrasting Works Based on Zeckendorf Representations

The results obtained using Zeckendorf representations in composition are infinitely variable, just as they are using any other mathematical sequence as a source for compositional material. Compositional decisions such as tempo and tuning, for example, have a dramatic effect on the outcome of such music. In order to better understand what kinds of different results can be achieved, we undertake an analysis of three compositions based on Zeckendorf representations. All of the compositions and scores discussed, including free audio and video files, are available at [2].

3.1. **Zeckendorf Representations no. 17.**

As the title indicates, 16 pieces based on Zeckendorf representations had preceded this work, the first of which was completed in early 2005 (B169 in [2]). Like the 16 works preceding it, $Z_i(n)$ was sonified as opposed to $Z_s(n)$. $Z_i(n)$ was mapped to two parameters: pitch and spatial location. This work (B1087 in [2]) exists in both 8-channel and stereo form. A total of $F_{14} = 377$ representations are sonified. It begins with the representation for 1, a requirement in the style for works based on Zeckendorf representations. Higher integers are represented by higher pitches, thus the orientation of the work is said to be ascending. Each Fibonacci number is attributed the dynamic level *forte* (loud). Since the sequence is mapped to the spatial domain, lower integers are heard as coming from the left and higher from the right; in the 8-channel version, the sound source moves clockwise in a circle as the Fibonacci indices j increase. VBAP, or vector-based amplitude panning, was used in the 8-channel version.

Delay between the right and left speakers – given in seconds – is used in the stereo version in addition to panning to simulate the location of the integers. A negative delay means that the sound is perceived as coming from the left, hence the left speaker plays .00069 seconds before the right for the first note of the composition, thereby creating the illusion that the sound is coming from the left. Attack and release – also given in seconds – refer to the time at the beginning and end of a single tone, in which the sound gets louder and quieter. Without attack and release, one might hear a click before and after each note. In this case, no additional attack was added to the dynamically compressed, low-pass filtered harpsichord sound; a release of .008 seconds was nonetheless added to the end of each note. All of the compositions discussed here were created in the software synthesizer Csound [13], originally developed at MIT by Barry Vercoe based on the family of software synthesizers known as MUSIC-N, first developed at Bell Laboratories by Max Matthews in 1957. Here is a slightly modified version of the web-based score for the work:

Collection VIII

Zeckendorf Representations no. 17
Casey Mongoven
November 9, 2008

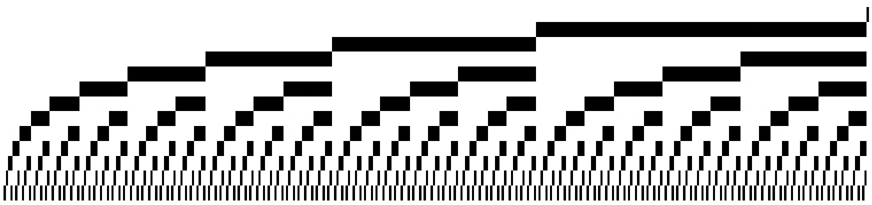
description of sequence: `Zeckendorf representation of n`
offset: `1`
number of members used: `377`
number of channels: `2 (or 8)`
piece length: `519.9961 seconds`
note value: `1.3793 seconds`
orientation: `ascending`
temperament: phi^7 + 1
lowest frequency: `175.45 Hz` **highest frequency:** `263.4 Hz`
number of unique frequencies: `13`
synthesis technique: `raw samples with analog dynamic compression, digital reverb and low-pass filter`
waves: Sperrhake harpsichord, keys Eb3 to Eb4 (retuned to phi^7 + 1)
dynamic: `each voice f`
simulated spatial location: `left to right through panning and delay (or clockwise with VBAP)`
delay: `-.00069 to .00069 (8-channel version without delay)`

attack: `natural`
release: `.008`
synthesis language used: `Csound`
note: `For the 8-channel version, the speakers should be arranged in a circle.`

Fibonacci index	location	frequency
2	1	175.45
3	2	181.49
4	3	187.74
5	4	194.21
6	5	200.89
7	6	207.81
8	7	214.97
9	8	222.37
10	9	230.03
11	10	237.96
12	11	246.15
13	12	254.63
14	13	263.40

sequence values used: `1, 10, 100, 101, 1000, 1001, 1010, 10000, ..., 101010101001, 101010101010, 1000000000000`

graph:



As indicated, this work used an acoustic instrument as an initial sound source: the composer's own Sperrhake harpsichord. Upon deciding to use the harpsichord, some technical considerations of the instrument came into play in composition: 1) it was desirable that each Fibonacci number be represented by a different frequency and a different string and 2) that the strings used be adjacent to one another. The tuning $\varphi^6+1 \approx 1.05573$ is the closest tuning satisfying φ^t+1 to the standard tuning of the harpsichord, which is – at least in theory[1] – based on $\sqrt[12]{2} \approx 1.05946$. Nonetheless, in order to avoid too much similarity to the standard tuning with this familiar instrument, the harpsichord was tuned to $\varphi^7 + 1 \approx 1.03444$ instead. This tuning imposed restrictions on how many adjacent strings could be retuned without breaking the strings that needed to be tuned to a higher frequency; in the end, it was decided that using 13 different tones representing 13 different integers would be feasible.

The duration for each representation in this work, 1.3793 seconds, is a speed of rendering at which one who is well practiced is still able to anticipate each note occurring in the representations. For such listeners, the properties discussed in 2.3.1 and 2.3.3 are very clearly perceived. Actually, the moderate tempo employed here represents a good middle-ground, and all properties discussed in 2.3 can be perceived in this work to varying degrees.

After calculating the number of occurrences of each Fibonacci number in the Zeckendorf representations from $Z(1)$ to $Z(377)$, each tone was recorded individually. The Fibonacci number 1, for example, was played and recorded 144 times – in reality a few times more, as the loudest and quietest notes were selectively removed from the raw recording. Following this, the order of the notes recorded was randomized using a true random number generator [15]; this randomization was carried out in order to smooth out natural tendencies the composer may have had in varying the amplitude of each attack in the course of playing, and in order to randomize any ever-so-slight variations in tuning which occurred over the course of the recording process.

It was the composer's goal in the sound design process to render the harpsichord practically unrecognizable, yet retain some of its natural qualities; to this end an analog dynamic compressor, digital reverb, and a digital low-pass Butterworth filter – with the filter cut-off sweeping exponentially from the 3rd harmonic partial of the fundamental frequency of the string used to the 8th over the course of each note – were applied. The effect of the low-pass filter in this work is most significant in transforming the sound of the harpsichord, as it acts to dampen the high frequencies present in the transient – the

[1]In reality, the tuning of a harpsichord is stretched due to a phenomenon known as inharmonicity, in which the actual harmonics of a physical string do not directly correspond to whole number multiples of their fundamental frequency [14].

initial attack – of the harpsichord, a feature which is so characteristic of its sound.

Each note of every chord had to be adjusted by ear individually in order to compensate for some auditory masking and loudness variations [16] which naturally occurred when combining the tones; the goal in this piece was to make each number represented sound at the dynamic level *forte*. To this end, both versions of the work were listened to dozens of times while the composer simultaneously adjusted the amplitude of each individual note.

The nature of the tuning used, $\varphi^7 + 1$, combined with the rich frequency spectra produced in the middle-range of the harpsichord and the decision to graph $Z_i(n)$ instead of $Z_s(n)$, result in a dense sound. The relatively slow speed of 1.3793 seconds per representation heightens the sense of anticipation of each chord.

3.2. **Zeckendorf Representations no. 18.**

A significant revision of the composer's notation was carried out in late 2009 and early 2010. Instead of the information in the scores being embedded in the HTML directly, a MySQL database was created to store such information and PHP is now utilized to extract the information and create the web pages for the scores dynamically. The most significant improvement was to the graph, created using the canvas element currently in development for HTML5; now when one hovers with the mouse over a certain number in the graph, the musical parameters attributed to that number are conveniently displayed.

In the following work, the values of $Z_i(n)$ are mapped to three musical parameters: pitch, loudness and location. In this work and the following, spatial location is given in degrees: -45° representing the left, 0° the middle and 45° the right.

Collection X

Zeckendorf Representations no. 18
Casey Mongoven
March 12, 2010

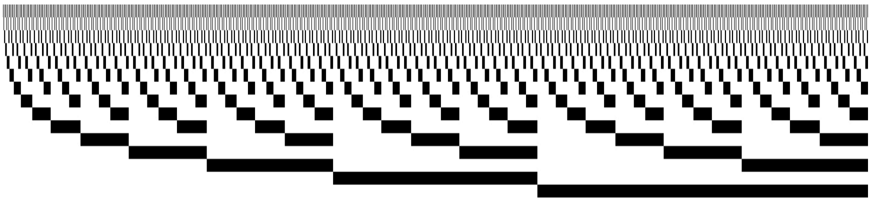
description of sequence: **MIN0102** `Zeckendorf representation of n`
classification of work: `audio-visual`
offset: `1`

number of members used: 1596
pitch orientation: descending
temperament: $\text{phi}^5 + 1$
number of channels: 2
note value: 0.25 seconds
piece length: 399 seconds
synthesis technique: STFT resynthesis and transposition with fast oscillator bank and low-pass filter
sound source used: red wine glass and Sperrhake harpsichord string tuned to same pitch recorded with laptop microphone and mixed
dynamic: p to ff
simulated location: right to left
delay: .00035 to -.00035 seconds
attack: .0055 seconds
release: .0089 seconds
programming languages used: Csound

Fibonacci index	frequency Hz	dynamic	simulated location degrees
2	1063.04387	p	45.0
3	975.117571	p+	40.6
4	894.463819	mp-	35.5
5	820.481085	mp-	29.7
6	752.617599	mp+	23.2
7	690.367225	mp+	16.0
8	633.265693	mf-	8.1
9	580.887133	mf	0.0
10	532.840899	mf+	-8.1
11	488.768657	f-	-16.0
12	448.341710	f-	-23.2
13	411.258550	f+	-29.7
14	377.242605	f+	-35.5
15	346.040181	ff-	-40.6
16	317.418567	ff	-45.0

1596 values used of sequence MIN0102 : 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, ..., 101010101010100, 101010101010101

graph of sequence MIN0102, 1596 values:



In this stereo work (B1171 in [2]), $F_{17} - 1 = 1596$ representations are heard; every bit string containing no consecutive 1s is heard up to length 15 (see 2.3.7). As in the last composition, the function $Z_i(n)$ was used. In this work, the speed of rendering is considerably faster than in the last: .25 seconds per representation. The property discussed in 2.3.1 is obscured slightly as a result; however, other properties such as those discussed in 2.3.2 and 2.3.3 are more clearly revealed.

In this work, natural sound sources were used again, this time purposely recorded using a cheap built-in laptop microphone – to additionally color the sound – as opposed to the high-end Neumann KM 183 microphone used in the previous work. The sound source for this work was a red wine glass struck with the knuckle on the back of the composer's right middle finger. This transparent, bell-like sound was then combined with the sound of a Sperrhake harpsichord string (A4) tuned to the same pitch of the glass, 429.4 Hz; in this mixture, the glass sound was given significantly more strength. The resulting mixed signal was then analyzed in the frequency domain using the short-time Fourier transform (STFT), and transposed, low-pass filtered versions of the tone were then synthesized – one transposition for each Fibonacci number needed.

The tuning used in *Zeckendorf Representations no. 18* contains very close to eight tones per octave. The unit interval $\varphi^5 + 1 \approx 149$ cents is located almost exactly in between the standard intervals of the tempered minor and major second, 150 cents.[2] Our familiarity with standard classical tuning therefore inevitably has a significant effect on our experience of this work; a bit sequence such as 1010101, for example, is hardly distinguishable from a tempered, fully diminished seventh chord in standard tuning. The property discussed in 2.3.6 is quite clearly perceivable in this work despite the faster tempo,

[2]The logarithmic unit of the cent is defined as $\sqrt[1200]{2}$, there are, therefore 1200 cents in the interval of the octave and 100 cents in a minor second.

due to the somewhat sparse nature of the tuning and the sonification of $Z_i(n)$ as opposed to $Z_s(n)$.

3.3. Zeckendorf Representations no. 19.

Collection X

Zeckendorf Representations no. 19
Casey Mongoven
March 19, 2010

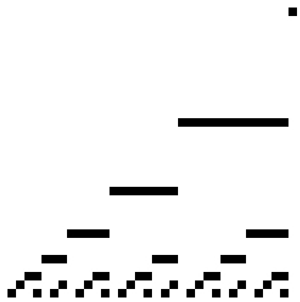
description of sequence: **MIN0102** `Zeckendorf representation on n`
classification of work: `audio-visual`
offset: `1`
number of members used: `34`
pitch orientation: `ascending`
temperament: phi^6 + 1
number of channels: `2`
note value: `6.69 seconds`
piece length: `227.46 seconds`
synthesis technique: `STFT resynthesis and transposition with fast oscillator bank`
sound source used: `red wine glass in studio recorded in Weimar, Germany; half-speed`
dynamic: `ppp to f`
simulated location: `left to right`
delay: `-.00035 to .00035 seconds`
attack: `.0377 seconds`
release: `.0521 seconds`
programming languages used: `Csound`

Fibonacci number	frequency Hz	dynamic	simulated location degrees
1	164.574500	ppp	-45.0
2	173.745923	ppp+	-43.2
3	183.428451	ppp+	-41.3
5	204.442344	pp-	-37.2
8	240.562032	pp+	-29.9
13	315.491359	p-	-15.3
21	486.859990	mp+	12.0
34	985.328598	f	45.0

34 values used of sequence **MIN0102** **:** 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, 10010, 10100, 10101, 100000, 100001, 100010, 100100, 100101, 101000, 101001, 101010,

1000000, 1000001, 1000010, 1000100, 1000101, 1001000,
1001001, 1001010, 1010000, 1010001, 1010010, 1010100,
1010101, 10000000

graph of sequence MIN0102, 34 values:



In this work (B1172 in [2]) in contrast to the previous two, $Z_s(n)$ is mapped to musical parameters as opposed to $Z_i(n)$. The parameters mapped to are pitch, loudness and location. As in the previous work, a (different) red wine glass was used as a sound source – in this case, the brim of the glass was rubbed with the composer's right index finger, which had just been washed with dish-washing soap in order to facilitate the excitation of the glass's higher resonant frequencies. As opposed to the percussive sound in the previous work, the resulting sound has a sustained quality similar to that achieved when playing such glasses with a violin bow. The glass sound was recorded in the composer's studio in Weimar in 2007 and retains to a great extent its original tone quality in the composition. Similarly to the previous work, transpositions of the original recording were created for each Fibonacci number using STFT.

For those familiar with the sound of interval relations based on the Fibonacci sequence, the result of mapping $Z_s(n)$ as opposed to $Z_i(n)$ is at once clear: the harmony of the entire piece is characterized by intervallic relationships derived from the Fibonacci sequence. In addition, the spacing (or location) of the sonified integers and their loudness is clearly derived from the Fibonacci sequence as well. The slow, meditative rendering of 6.69 seconds per representation obscures the rabbit sequence and the recursive property discussed in 2.3.3 and 2.3.5 to some extent. In contrast, the properties discussed in 2.3.1 and 2.3.6 are much more clearly illuminated.

4. The Graphic Element

A synchronized visualization of the Zeckendorf representations was produced in the programming language *Processing* [17] for each work presented here, in which each representation appears as it is heard in a point graph. These works can, however, also be experienced without graphics. The intention of the graphic element is to enhance and reinforce the perceptual clarity of the audio – to serve as an aid in the sonic illustration of the properties discussed above, for example. In order to better serve this purpose, a minimalistic approach similar to that taken for the music was taken to the graphic element. Whereas in the audio one cannot "look back" at the representations which have occurred, the visual element does not have this restriction, provided one can fit all the representations which have occurred on the screen. The residual visualization of past representations can aid in the sonic recollection of what has transpired. In compositions based on Zeckendorf representations, a counter is placed at the top, indicating the integer currently being represented along with the Fibonacci numbers summing to that integer. Here is a modified still image of the penultimate representation in *Zeckendorf Representations no. 19*:

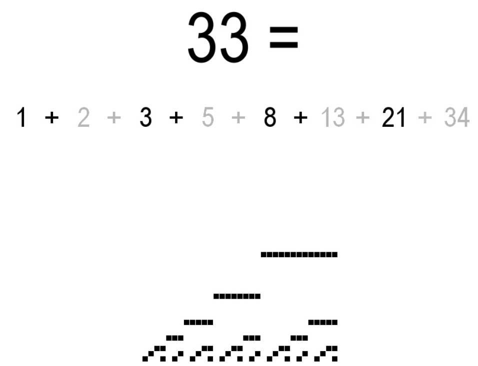


Figure 5

5. Other Compositions

Interesting results can be achieved in this style with sequences derived from $Z(n)$ as well; $P_h(n)$ can, for example, be derived from $Z(n)$. Other sequences of interest are the number of 1s (A007895) or 0s (A102364) present in $Z_b(n)$, and number of runs of equal bits in $Z_b(n)$ (A104324). See compositions B1154, B827 and B1143 in [2].

The composer has also applied this principle to the closely related base phi (phinary) representations [18] (see B416 in [2]). Work in progress includes the Lucas base representations and representations based on other generalized Fibonacci sequences.

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