A STYLE OF MUSIC CHARACTERIZED BY FIBONACCI NUMBERS AND THE GOLDEN RATIO

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Many composers of the 20th century used Fibonacci numbers and the golden ratio in their works. None of these composers, however, made them the basis of a style. Creating a style of music characterized by Fibonacci numbers and the golden ratio requires discarding musical traditions and rethinking stylistic elements from the ground up. In this style, mathematical properties of sequences related to the golden ratio and Fibonacci numbers are converted into musical properties. This paper introduces the style, presenting three short works based on Fibonacci-related sequences.

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The purpose of this paper is to introduce a style of music which has its roots in the Fibonacci numbers and the golden ratio. Although many composers of the past used these mathematical objects in their compositions, none made it the basis of their musical style. Instead, these composers superimposed the Fibonacci numbers and the golden ratio on top of their own style – in most cases in order to provide a robust formal structure.

1. SHORT HISTORY OF FIBONACCI IN MUSIC

The first references to the golden ratio in connection with music appear to stem from German philosopher Adolf Zeising [1], who referred to the proportion in 1856 as:

> "that standard proportion which underlies the arrangement of the human form, the construction of the more beautiful animals, the construction of plants, namely in the form of their leaf-arrangement, the forms of various crystals, the arrangement of the planets, the proportions of architectural and sculptural works recognized as being the most beautiful, the most satisfying chords in musical harmony, as well as many other things in nature and art."

Although Zeising's hypothesis aroused great interest in the field of aesthetics, it was confronted with great skepticism by the scientific community. This is evident in the writings of Gustav Theodor Fechner [2], who wrote in 1870:

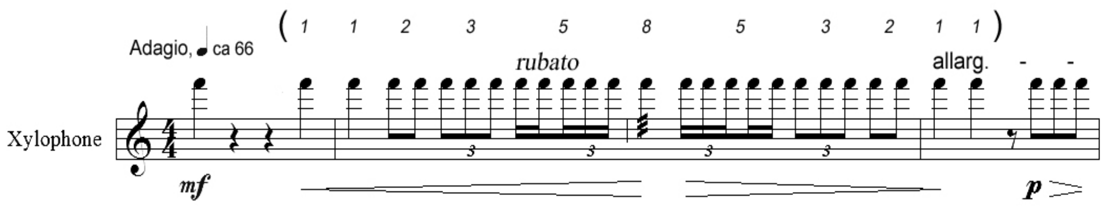
> "Because he [Adolf Zeising] is of the opinion that the same principle of pleasantness must be valid in both the field of vision and sound – and does not want to hear of anything at all that deprives golden ratio of power – he finds himself obligated, in clear contradiction to his general judgment, to declare both [major and minor] sixths to be the most musically pleasant proportions, for which he believes to find reasons in musical compositional proportions. As a result, he would clearly be forced instead of the pure sixth, to declare the impure sixth which corresponds *exactly* to the golden ratio, to be the musically most pleasant proportion…"

One can see the widespread influence that Zeising's ideas had on classical music theory when one looks at the writings of Arnold Schönberg [3], who wrote in his *Harmonielehre* (1911) in reference to musical form:

> "I don't believe in the golden ratio. At least, I don't believe that it is the single formal principle for our sense of beauty; rather at most one among many, among countless many."

The tone of this statement makes it clear that the idea of the golden ratio had been discussed among music theorists for some time before Schönberg wrote this.

Although composer Béla Bartók is not known to have discussed his use of the golden ratio and Fibonacci numbers, it appears likely that he consciously used them in his *Music for Strings, Percussion and Celesta* (1936) [4]. Theorist Ernõ Lendvai [5] is credited with this discovery. My own independent analysis has confirmed that Bartók timed the climax of the first movement to occur extremely close to the golden section – within about 1/1000 of the proportion, depending on the tempi chosen (my own analysis used the average of the recommended tempi). This theory is further supported by Bartók's apparent use of a palindromic rhythm based on Fibonacci numbers in the opening of the third movement:



It would be a great coincidence if Bartók did not consciously implement the sequence in this passage.

American composer Joseph Schillinger [6] suggested creating pitch structures based on the Fibonacci sequence in his series of books *The Schillinger System of Musical Composition* (1941). Iannis Xenakis [7] used Fibonacci numbers in his *Anastenaria, Le sacrifice* (1953). Luigi Nono [8] used successive note-lengths derived from the Fibonacci sequence in *Il canto sospeso* (1956). György Ligeti [9] acknowledged his use of the golden ratio in determining the lengths of the respective sections in *Apparitions* (1958-59). Ernst Krenek [10] composed a work entitled *Fibonacci Mobile* (1964), in which he utilized the series in the construction of the proportions. Brian Ferneyhough [11] writes of his "application, in various roles, of the proportions inherent in the Fibonacci series" in his *Time and Motion Study I* (1971-77). Karlheinz Stockhausen [12], in a 2004 interview with Piergiorgio Odifreddi, stated that he used the Fibonacci numbers in many of his works, for the "duration of different sections as well as for the density of sound-masses."

2. THE ORIGINS OF THE STYLE

In the music of the composers mentioned above, the Fibonacci numbers and golden ratio did not form the basis of a musical style. These were isolated works, and in most of these composers' works, no trace of their use is evident.

In 1997, a great curiosity began to drive my work; I wanted to know what a music would sound like in which Fibonacci numbers and the golden ratio played the central role. In my first compositions with Fibonacci numbers, I used them as a means of providing structural and tonal cohesion, similar to the composers discussed above. These first compositions were all written in 12-tone equal temperament (using the notes of the piano), and in most of them, I used the Fibonacci and Lucas sequences modulo 12. A few were written in 24-tone equal temperament using modulo 24. In 2001, I decided that I was no longer interested in using a tuning system handed down to me through tradition; after all, that tuning system had nothing to do with Fibonacci. I set out to create my own tuning system.

3. A SYSTEM OF TUNINGS BASED ON FIBONACCI AND THE GOLDEN RATIO

The base interval r in 12-tone equal temperament, the half-step, is derived from

$$r = \sqrt[12]{2} \approx 1.059463.$$

So if the note A on the piano is tuned to 440 Hz, that means the next note up, A#, is

$$440(\sqrt[12]{2}) \approx 466.2 \text{ Hz}.$$

The cent serves as the logarithmic unit of measure for musical intervals, and it is defined that

$$\sqrt[12]{2} = 100 \text{ cents};$$

there are therefore 1200 cents in one octave (the interval of an octave is created by a 2:1 frequency ratio). What happens if we use the golden ratio as our base interval instead?

$$r = \Phi = \frac{1+\sqrt{5}}{2} \approx 833 \text{ cents}$$

This is a rather large interval, lying approximately one third between the minor and major sixth. The human hearing range spans from approximately 16 to 20000 Hz. How many pitches is it possible to have in our hearing range using the golden ratio as a base interval?

1. 16 Hz
2. 25.9 Hz
⋮
15. 13488.0 Hz

If we start at 16 Hz, the highest note would be approximately 13488 Hz, and we would have a total of only 15 possible pitches. This is an interesting tuning, but it is not very economical in terms of pitch space. Let us try using the smaller golden ratio value φ = 1/Φ and

$$r = \varphi^2 + 1 \approx 1.381966 \approx 560 \text{ cents}.$$

How many notes are possible now?

1. 16 Hz
2. 22.1 Hz
⋮
23. 19727.0 Hz

This tuning also has rather large intervals. Let us create a system of temperaments based on

$$r = \varphi^x + 1,$$

where *x* is a positive integer, noting that [13]

$$\lim_{n\to\infty} \frac{F_n + F_{n-x}}{F_n} = \varphi^x + 1.$$

Now we have a system of tunings based on the golden ratio and Fibonacci numbers.

	base interval approx. size in cents	pitches in hearing range	notes within one octave
$r = \Phi = \varphi + 1$	833	15	2
$r = \varphi^2 + 1$	560	23	3
$r = \varphi^3 + 1$	367	34	4
$r = \varphi^4 + 1$	236	53	6
$r = \varphi^5 + 1$	149	83	9
$r = \varphi^6 + 1$	94	132	13
$r = \varphi^7 + 1$	59	211	21
$r = \varphi^8 + 1$	36	339	33
$r = \varphi^9 + 1$	23	546	54
$r = \varphi^{10} + 1$	14	881	86
$r = \varphi^{11} + 1$	9	1423	139
$r = \varphi^{12} + 1$	5	2300	224
$r = \varphi^{13} + 1$	3	3719	362
$r = \varphi^{14} + 1$	2	6015	585
$r = \varphi^{15} + 1$	1	9731	946

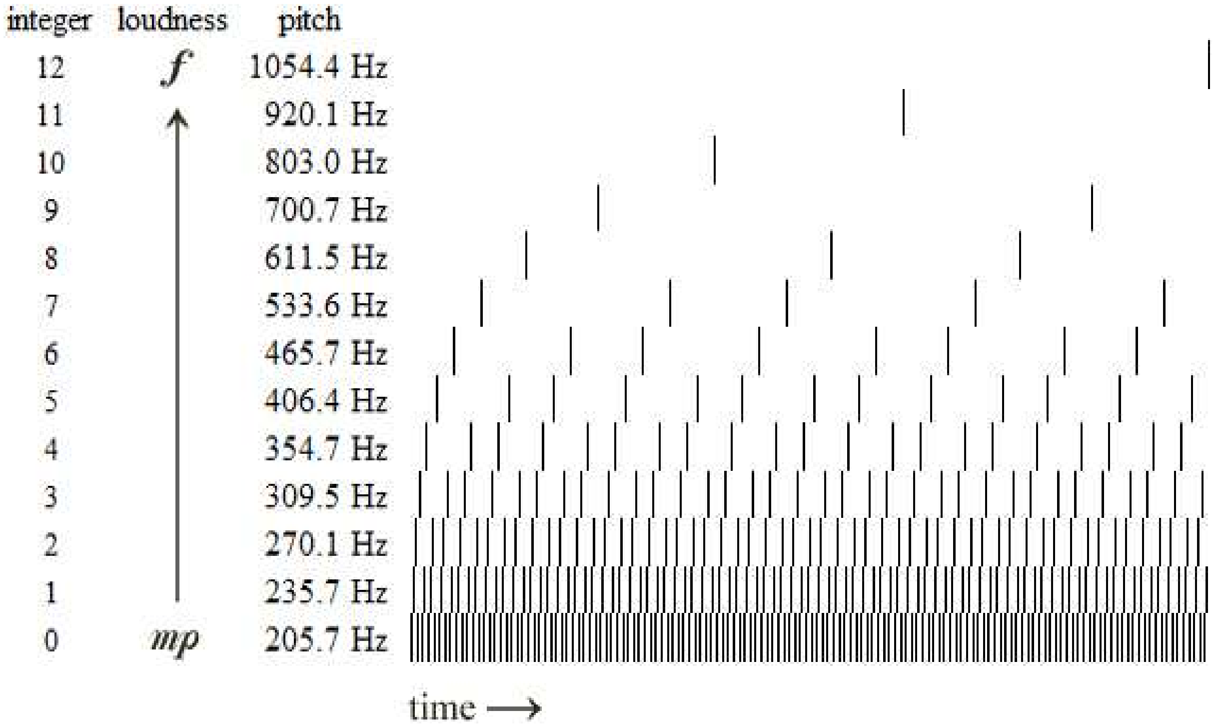
These tunings contain no octaves – or any other pure interval for that matter.

4. PURIFICATION OF THE STYLE

Having created a system of temperaments, I decided that a style of music characterized by Fibonacci numbers and the golden ratio had to be comprehensible above all else. By this is meant that there could not be any extra elements used in a composition that might obscure the mathematical object which that composition was based on; it had to be crystal clear how the music related to the mathematical object it portrayed. Rather than becoming more complicated, as the styles of most composers tend to do, my style became simpler in the most fundamental sense. All elements of tradition in my music were abandoned. I wrote the first work in this style in April 2002, using the software synthesis language Csound [14].

5. CHARACTERISTICS OF THE STYLE, COMPOSITIONAL METHODS

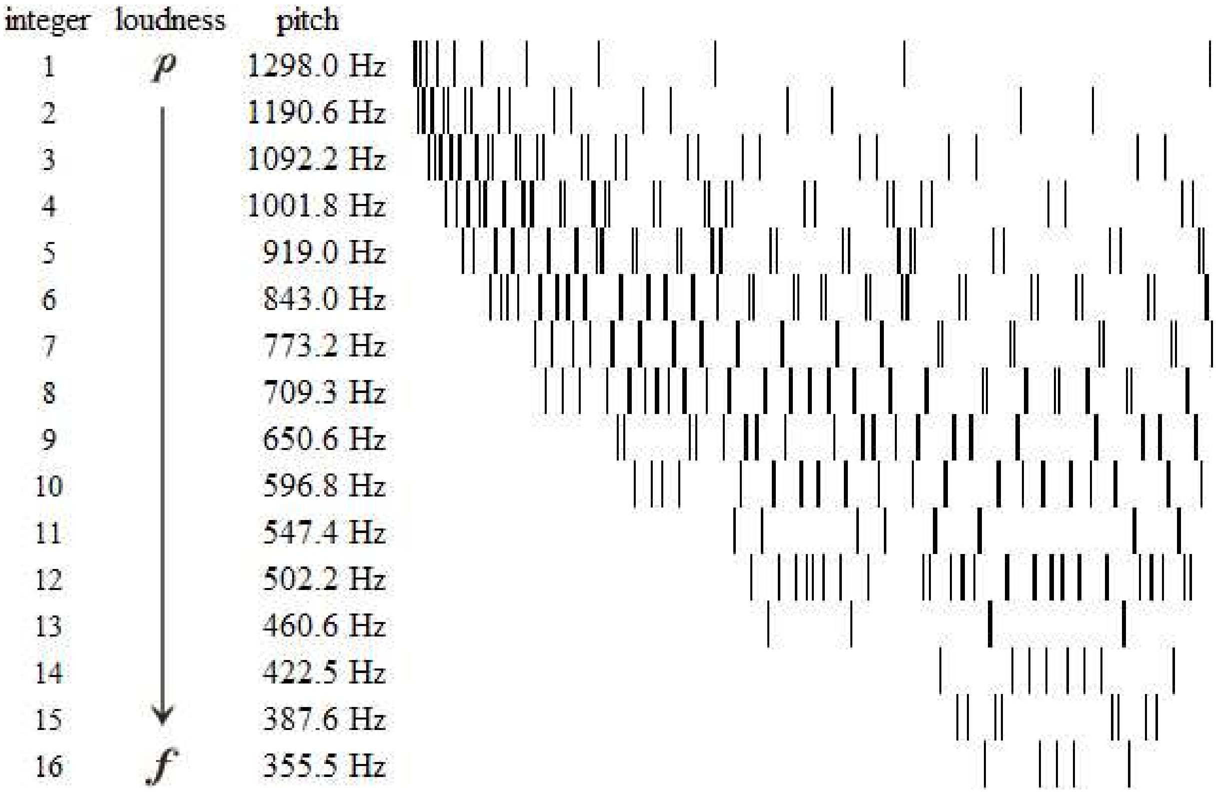
For each work in the style, a sequence related to the Fibonacci numbers and the golden ratio is selected. Generally, only one sequence is used in a composition. Similarly, only one temperament is used in a composition, selected from those described earlier. The sequence is then "graphed" into musical space in a direct and simple manner. By this is meant that each integer occurring in the sequence is assigned a pitch; integers are often assigned other musical parameters too, such as loudness, timbre and spatial location. In most cases, the integers are heard one after another, ordered according to their indices. To provide a simple example of what is meant here, let us look at a graph of my work *Horizontal Para-Fibonacci Sequence no. 31*; the original score and sound files are available at [15]. This work is based, as the name indicates, on the horizontal para-Fibonacci sequence (A035614 in The On-Line Encyclopedia of Integer Sequences – or OEIS [16]), which gives the column number in which integers occur in the Wythoff array.



Each point in the graph represents a musical note. Each note receives an equal duration, in this case 50 milliseconds. This duration is referred to as the *note value*. In this work, the temperament based on $\varphi^4 + 1$was used. The lowest integer, 0, is given a dynamic value of mezzopiano (medium-quiet); the highest integer is given a dynamic level of forte (loud). The arrow pointing upwards in between these two dynamic levels represents a gradual increase in loudness from integers 0 to 12 (from mezzopiano to forte). 377 integers are used in this composition – a Fibonacci number. We can see from the graph that every time a certain integer occurs in the sequence, it sounds the same; this is critical for transforming the mathematical properties into musical properties in a direct and perceptible manner.

Every time the integer 0 occurs in the sequence, the pitch heard is 205.7 Hz at a dynamic level of mezzopiano. The choice of temperament, note value and dynamic level is a purely artistic decision.

If the pitch rises with the value of an integer, as in the previous example, the orientation of the work is termed *ascending*. If the orientation in a work is descending, the pitch falls as the value of the integer increases. The following graph is from my piece *S(n) Rep Sequence no. 20*; the sequence used here (A000119 in the OEIS) appears under that name in *Fibonacci and Related Number Theoretical Tables* [17]. This sequence gives the number of possible representations of integers as sums of distinct elements of the Fibonacci sequence beginning 1, 2, 3, 5, 8, ... This is an example of descending orientation. The original score and sound files are available at [18].



In this work, the temperament based on $\varphi^5 + 1$was used. The note value is 58 milliseconds.

6. NOTATION

When I made the decision to abandon traditional elements of classical music and create my own system of tunings, it became clear that classical notation would no longer be suitable for my style.

My notation has been developed and refined since I created the first works in this style. My current notation is written in HTML. The graphs above are not from the original notation, and were created specifically for explanatory purposes. In the actual scores, the parameters (meaning the pitches, dynamic level and other information seen on the left here) are on a separate page from the graph, and more information on the compositions is provided.

The following score has been slightly modified from its original HTML form for explanation and practical purposes. The lists have also been truncated (indicated by triple-dots) to save space. The original score and sound files can be found at [19].

Collection VI

Fibonacci Entry Points no. 24
Casey Mongoven
March 27, 2008

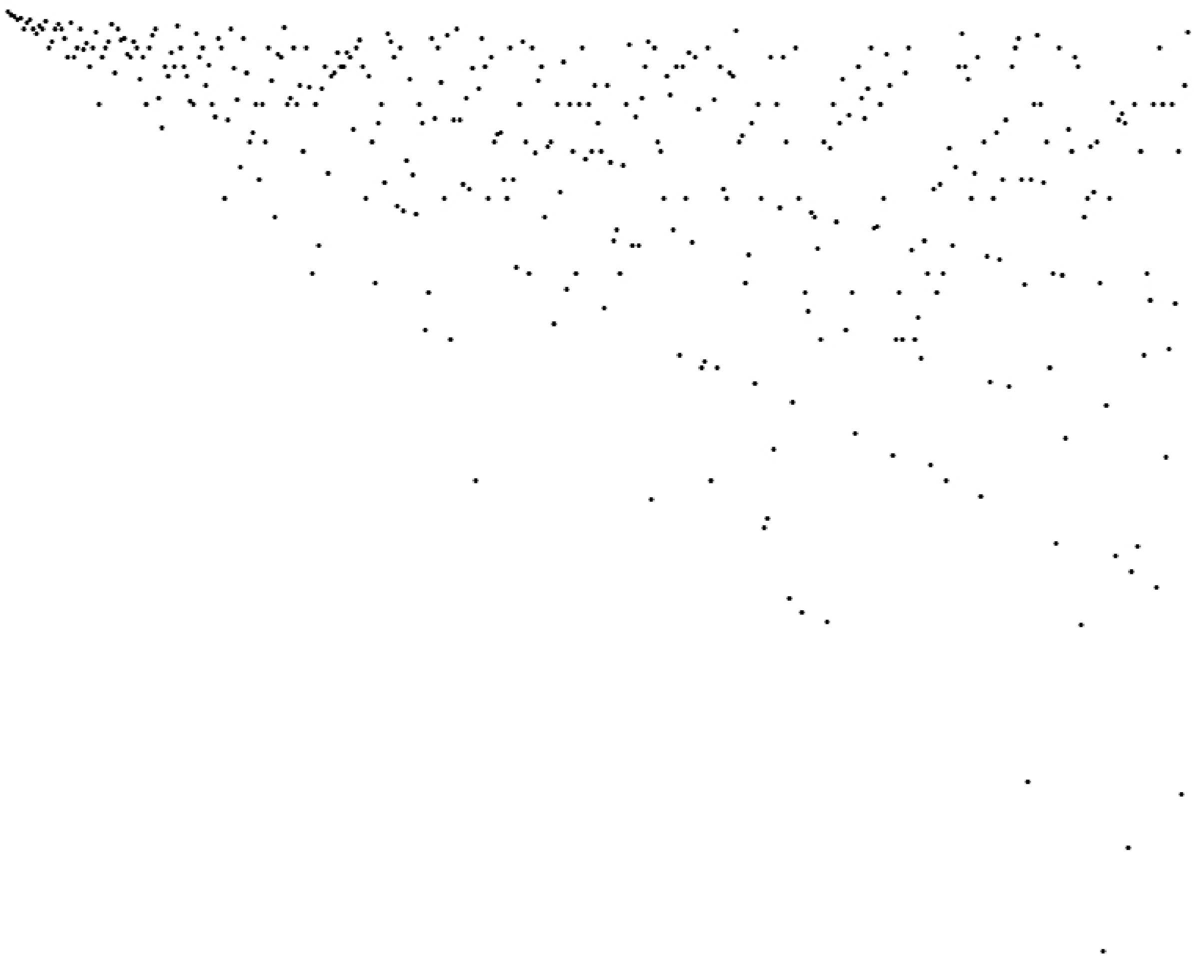
description of sequence: index of lowest Fibonacci number that n divides[1]
offset: 1[2]
number of members used: 377
number of channels: 2[3]
piece length: 20.358 seconds
note value: .054 seconds
orientation: descending
temperament: phi^12 + 1
lowest frequency: 96.5591688 Hz **highest frequency:** 618.65 Hz
number of unique frequencies: 140[4]
synthesis technique: wavetable[5]
wave: wave one to wave two[6]
wave one source: large balloon[7]
wave two source: electric drill[8]
dynamic: mp to f
simulated spatial location: right to left[9]
delay: .0018 to -.0018[10]
attack: .0063 to .0055 (roll-off .0000013)[11]
release: .0102 to .0089 (roll-off .0000022)
synthesis language used: Csound

1) A001177 2) index of first term used 3) a stereo work 4) number of distinct integers (and therefore frequencies, loudness levels, etc.) in first 377 members of this sequence 5) Wavetable synthesis involves using recorded sounds and "stretching" them or compressing them to create different pitches. 6) The timbre changes according to the sequence, just as the loudness and pitch do, from wave one to wave two. 7) The sound of a balloon letting out air was use as a sound source for wave one. 8) The sound source of wave two was an electric drill. 9) Using psycho-acoustic phenomena or "stereo effect," each integer is given its own simulated spatial location, so the spatial location of the sound is also governed by the sequence. 10) Delay between the right and left speakers – given in seconds here – is one of the elements used to create the spatial element. 11) Attack and release – given in seconds – refer to the time at the beginning and end of a single tone, in which the sound gets louder and quieter. Without attack and release, one would hear a click before and after each note. The attack and release values are controlled by the sequence here, so that higher integers sound crisper than lower ones.

integer	frequency	wave	dynamic	s.s.l.	attack	release
1	618.6500	w1*.778+w2*.222	mp	right	.006300	.010200
3	614.8252	↓	↓	↓	.006297	.010195
4	612.9217	↓	↓	↓	.006295	.010193
5	611.0241	↓	↓	↓	.006294	.010191
6	609.1324	↓	↓	↓	.006293	.010189
⋮	⋮	⋮	⋮	⋮	⋮	⋮
500	131.6618	↓	↓	↓	.005633	.009117
534	118.4877	↓	↓	↓	.005588	.009043
600	96.55917	w1*.206+w2*.794	f	left	.005500	.008900

sequence values used: 1, 3, 4, 6, 5, 12, 8, 6, 12, 15, 10, 12, 7, 24, 20, 12, 9, 12, 18, 30, 8, 30, 24, 12, 25, 21, 36, 24, 14, 60, 30, 24, 20, 9, ..., 72, 534, 358, 60, 342, 90, 220, 168, 185, 60, 368, 24, 60, 285, 216, 60, 187, 90, 500, 48, 14

graph:



7. MULTIMEDIA WORKS

At the end of 2007, I realized a goal I had had for some time – to create a multimedia work in this style. Using the graphic programming language Processing, I created a presentation of 13 works in which point graphs were showed synchronized with the music. This means that when a tone is played, it appears on the screen at exactly that moment. A somewhat minimalist style was used for the graphic element; the tones appeared in white on a black background while they were being played, fading to grey once they had been heard. The response from the audience was quite positive, and I found that people had a better understanding of what was happening in the music.

8. CONCLUDING WORDS

With this music, I hope to answer the age-old question of what a musical style sounds like which has its roots in the Fibonacci numbers and the golden ratio. Many more numeric sequences related to mathematics of Fibonacci numbers and golden ratio are used in my music. These include among others; Zeckendorf representations (A014417), Wythoff and Stolarsky arrays [20], base Phi [21] and base Lucas representations (A130310 and A130311), signature sequences based on Phi (A084532, A084531 and A118276), and Pisano periods (A001175). Catalogue entries on my website always contain sound files, scores and links to the sequence used in the OEIS. To hear more works in this style, visit http://caseymongoven.com; I offer the public free access to my music.

Special thanks goes to Ron Knott for his support over the years and comments while I was writing this paper.

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